DANIEL MICHELSON

The Shape of the Menorah

The shape of the Menorah is reconstructed mathematically, using the available data from the Torah and Talmud with some additional natural constraints. The crucial piece of information is the total weight: 3000 *shekels*. As a result, three different shapes of the Menorah are obtained: straight, round, and parabolic. The fact that they all have almost the correct weight, despite very few choices of parameters, and the fact that the fixtures have simple, rational dimensions, should puzzle a skeptical mind.

INTRODUCTION

Of all the vessels in the Tabernacle and the Temple, the Menorah appears to have been the most complicated one. Our sages say (Bamidbar Raba 15:10), that it was difficult for Moses to understand and remember how to make the Menorah, until Bezalel constructed it. Another opinion (Tanchuma, Shemini 8) states that, even after all the explanations, Moses still was unable to make the Menorah, until God told him to throw the gold into fire—and the Menorah was made by itself. Thus, there were both "theoretical" and "practical" difficulties in manufacturing the Menorah. Everyone is familiar with the bulky Menorah that can be seen on the Arch of Titus in Rome. We will show that the true Menorah was much "leaner." The aim of this article is to reconstruct mathematically the true structure of the Menorah, with all its components and fixtures. A crucial piece of information is the weight of the Menorah. We will see how this single constraint helps to reveal all the necessary details of the Menorah. Since the weight of the Menorah is given in *shekels*, we include in this paper a discussion of the weight of the *shekel*, and its relation to the units of volume.

1. WHAT IS KNOWN ABOUT THE MENORAH?

It is written in the Torah: "And thou shalt make a Menorah of pure gold: of beaten work shall the Menorah be made: its shaft, and its branches, its bowls, its bulbs, and its flowers, shall be of the same" (Exodus 25:31). The nine verses that follow provide additional details. Even with this description, however, it is impossible to reconstruct the Menorah. Thus, we do not know the precise shape of the bowls, bulbs, and flowers. Moreover, there is a major controversy regarding the shape of the branches. According to Rambam and seemingly Rashi, the branches were straight. According to Even Ezra, they were round. Controversy also surrounds the shape of the legs. According to Rashi, there was a box-like base with three legs at its lower end. In a picture drawn by Rambam, it appears that the base was dome-like, and stood on three legs. The Talmud (Menachot 28b) presents additional data relating to the height of the Menorah and its components:

> The height of the Menorah is 18 palms. The legs and the flower 3 palms; 2 palms plain; a palm with bowl, bulb and flower; 2 palms plain; a palm with bulb and two branches going out of it—one there and another there, extending and rising against the height of the Menorah; a palm plain; a palm with bulb and two branches etc.; a palm plain; a palm with bulb and two branches etc.; 2 palms plain; there are left three palms with 3 bowls, bulb and flower.

The width of the Menorah is not explicitly stated. However, since the Menorah stood against the length of the table, which was 12 palms, one can infer that the width of the exterior branches was 12 palms. It seems natural to divide this width into six equal parts, so that the distance between the upper ends of neighboring branches is two palms. Yet, there is one piece of information that is crucial in reconstructing the shape of the Menorah: namely, its weight. "Of a talent of pure gold shall he make it, with all the vessels" (Exodus 25:39). The talent of the Torah is 3000 *shekel*s, while the *shekel* is either approximately 14 gr (according to Rashi), or approximately 17 gr (according to Rambam and the *geonim*). The average density of pure hammered gold is 19.3 gr/cm^3. Based on these figures, one can calculate the volume of the Menorah. It was either, approximately, 2176 cm^3 or 2642 cm^3. This is a relatively small volume. The kind of Menorah portrayed on the Arch of Titus in Rome, or exhibited today in the Cardo in Jerusalem, is several times more massive. One may suggest that the branches of the Menorah were hollow. But this seems impossible, since the Menorah was hammered out of a single piece of gold. A heavy box-like base is also excluded. On the other hand, the branches could not have been thin, since pure gold will bend easily. It seems that the design of the

Menorah was optimal, with no "extras." Our hypothesis was that the single weight restriction would provide the necessary clue for the several unknown parameters relating to the form of the Menorah. Before proceeding, we should establish the exact relation between the weight of the *shekel* and the unit of length—the palm.

2. THE CUBIT AND *SHEKEL* OF THE TORAH

It is impossible in this short article to fully discuss the Torah measures and weights. Our premise is that the Torah cubit is 48 cm. This is the average length of the arm in our time, and the accepted halakhic length, attributed to Rabbi Haim Nae. (There is another widespread opinion, supported by Hazon Ish, according to which the cubit is 6/5 longer).

As for the weight of the *shekel*, there is disagreement between Rashi, on the one hand, and Rambam and the *geonim* on the other. According to Rashi, the *shekel* was about 14 gr, and according to Rambam and the *geonim* it was about 17 gr. The actual *shekel* coins from the time of the destruction of the Second Temple and from the time of Bar-Kokhba's revolt weigh about 14 gr. There was, however, another standard of the *shekel*, the so-called Attic *tetradrachma* of 17.28 gr. According to Josephus in *Antiquities* 3, 8, two *shekel* of Moses was the equivalent of four Athenian *drachmae*. It turns out that the weight of this *shekel* is related to the basic unit of volume—a cup. One cup equals *seah*/96, where 40 *seah* are equal to 3 cubic cubits. With a cubit of 48 cm we get a cup = 86.4 cm^3. Thus, one cup of water weighs exactly 5 *shekels* of the Torah. Perhaps this was hinted at by Joseph when he placed the silver cup in Benjamin's bag. Joseph was sold for 20 silver *dinars* or 5 *shekels* (see *Yerushalmi*, Shekalim 9b). In the article "The Cubit and Shekel of Torah" at the site www.truthofland.co.il many arguments are brought to support this ratio between the unit of weight and the unit of volume. In the sequel, this ratio will be taken for granted.

Notice that our computation of the volume in terms of the units of length does not depend on the exact length of the cubit and the exact weight of the *shekel*, but only on the ratio between the unit of the volume and the *shekel*. There was certainly a rational relationship between the two (e.g., according to Rashi, a *lug* of four caps of water weighs a *mane* of 25 *shekels* of the Second Temple). Thus, if a reasonably accurate estimate of the cubit and the *shekel* lead to a simple integer relationship, we can postulate this relationship with absolute accuracy. For convenience, we carried out the calculation in centimeters. These could be replaced everywhere by 1/48 of a cubit.

Another factor that affects the calculation is the specific gravity of pure hammered gold at room temperature. Gmelin, *Handbuch der anorganischen Chemie*, v. 62, p. 451, quotes several measurements in the range of 19.2-19.4 gr/ cm^3. Based on these measurements, he fixes the value of 19.3 as the most practical and useful one. Some tables give the value 19.32. We postulated the 3 digit approximation 19.3. Our calculations were carried out with much greater accuracy. The final result, as will be shown in the sequel, accords with the total weight up to 0.03 of a *shekel* for the straight Menorah and 0.01 *shekel* for the round Menorah (with no parameters to play with). The reader may regard this as mere coincidence (with a probability of about 3×10^{-5} compared with the accuracy for the density of 19.32) or the result of the author's inserting hidden parameters. In the latter case, the reader is challenged to insert such parameters in a natural way to attain the same accuracy for the density 19.32. For convenience, we have collected all the assumptions in Section 15.

3. THE APPROXIMATE CALCULATION OF THE MENORAH

Our calculation will begin with an estimate of the weight of the body of the Menorah: the main stem, the side branches, and the legs. The thickness of these parts is not found in the Bible or later sources. The basic unit of length below the palm is a thumb, 1/4 of a palm, i.e. 2 cm. This was actually the smallest unit for vessels and Tabernacle mentioned in Talmud (the thickness of the walls of the ark, according to Rabbi Yehuda in Bava Batra 14a and the thickness of the upper rim of the boards—again according to Rabbi Yehuda in Shabat 98b). Thus, we will start with the assumption that the stem, the branches, and the legs were thumb thick. (Remark: We later found this supposition, with the same arguments regarding the stem and the branches, in the book *Mikdash Aharon* by Rabbi Aharon Zvi Even Chen.) According to Talmud Menachot 28a, as quoted in Section 1, the Menorah was 18 palms high. The three legs separated from the stem at h = 3 palms. (Remark: it is possible that one should subtract from h = 3 the height of the flower.) The first pair of branches separated at h = 9, at the middle of the height of the Menorah, the next pair two palms higher, at h = 11, and the last at h = 13 palms (see the explanation in the Appendix, Assumption 11). The width of the Menorah is not stated explicitly. Since the Menorah was standing in front of the table, it is reasonable to assume that its width was 12 palms—like the length of the table. Another natural assumption is that the gaps between the seven lamps were all equal. Hence, the span of the lower branches was 6 palms, of the middle ones 4 palms, and of the upper ones 2 palms. The span of the legs is not known but we will assume that it was 2 palms (from the stem), like the span of the upper branches. According to Rambam, all the branches were straight. We will therefore assume that the legs were also straight. Thus, the total length of branches and legs is:

$$(3.1)\quad 3\sqrt{(3^2+2^2)}+2\sqrt{(9^2+6^2)}+2\sqrt{(7^2+4^2)}+2\sqrt{(5^2+2^2)}=59.345$$

The length of the stem is 18-3 = 15 palms. Hence, the total length is 74.34 palms. The cross section of all parts is a circle of diameter 2 cm. Hence, the volume of the body of the Menorah is 74.34 x 8π = 1868 cm^3, and its weight is 1868 x 19.3/17.28 = 2086 *shekel*s. The remainder, of about 914 shekels, should be divided between 49 fixtures – 22 bowls, 11 bulbs, 9 flowers, and 7 lamps. If we were to make all these 49 details equal, their weight would be approximately 18.6 *shekel*s. It would be pleasing if the weight were exactly a *mane* of 20 *shekel*s. The lamps also had handles to support the wicks. If we assume that their total weight was *also* 20 *shekels*, then the body would weigh 100 *mane* and the remainder 50 *mane*. This would be an esthetically pleasing solution. However, the weight of the body we obtained is apparently minimal! It came to my mind that the number π in the Talmud was always counted as 3. (Remark: In the article "Number π and Solomon's Molten Sea" at the site www.truthofland.co.il the reason for this approximation of π is discussed.) If we replace π by 3, the weight of the body would be approximately 1993 *shekel*s, close to the required 2000.

How can one make π equal 3? By assuming that the cross section of the branches and legs was not a circle with radius 1 cm, but a perfect dodecagon (12-sided polygon) inscribed in this circle! Then, the area of the cross section is exactly 3 cm^2, while the exterior diameter of the cross section is still a thumb.

Of course, our simplified calculation of the volume of the body is not exact. The side branches do not originate from the center but from the surface of the bulb. Hence, they are shorter than we calculated. The legs too have mutual intersection. It will turn out, in Section 9, that the actual weight of the body of the Menorah, with a circular cross section of radius 1, is exactly 2000 *shekel*s. Yet, the idea of the dodecagon cross section implied by the simplified calculation will be implemented in a Menorah with circular and parabolic branches. The main lesson we learned is that one should separate the body from the fixtures so that each fixture weighs 20 shekels. We will now proceed with the fixtures.

4. THE BULB

According to Talmud Menachot 28b, the bulbs look like Cretan apples. Their length is a palm, since the Talmud says "a palm with bulb and two branches going out of it." In pictures, Menorah bulbs are usually shown as a kind of ellipsoid. We will define it precisely as an ellipsoid of rotation around the axis of the Menorah, cut by two horizontal planes at a distance of 4 cm from its center (see Fig. 1). Thus, its length is exactly 8 cm. The weight of the ellipsoid should be 20 *shekels* of 17.28 gr, plus the weight of the stem of length 8. With r denoting the horizontal distance from the axis and z the vertical coordinate measured from the center of the bulb, the equation of the ellipsoid is:

$$(4.1)\quad r^2/a^2+z^2/b^2=1;\ \text{where}\ 1/a^2+4^2/b^2=1$$

The volume of the ellipsoid between two cuts is:

$$(4.2)\quad V=\pi\int a^2(1-z^2/b^2)dz,\quad -4\le y\le 4;\quad V=2\pi(4a^2(1-16/b^2/3))$$

On the other hand, this volume is equal to the volume of a segment of the stem of length 8 and cross section 3, plus the volume of 20 *shekels* of gold. Thus we obtain the equation:

$$(4.3)\quad V=2\pi(4a^2(1-16/b^2/3))=8\times 3+20\times 17.28/19.3=41.9067\ \text{cm}^3$$

and:

$$(4.4)\quad a=\sqrt{1.5(V/8/\pi-1/3))};\quad b=4\sqrt{(1-(1/a)^2)}$$

$$(4.5)\quad a=1.41461\ldots;\quad b=5.65526\ldots$$

Indeed, the "apple" is quite narrow. Notice that $a\approx\sqrt{2}$, and $b\approx 4\sqrt{2}$ (the last follows from the first). The slope of the bulb at its base is thus very close to 4.

We will also calculate the axis of the bulb in the case that the cross section of the branches is a circle with radius 1. Then the volume V is:

$$(4.6)\quad V=8\pi+20\times 17.28/19.3=43.0395$$

From equations (4.1) and (4.2) we obtain:

$$(4.7)\quad a=1.4383,\ b=5.5652\ \text{cm}.$$

5. THE BOWL

The Talmud in Menachot 28b states that the bowls look like the ones from Alexandria. Rashi explains that they are long and narrow. In the picture drawn by Rambam, they look like cones with a narrow base. We will therefore make the bowl like a cone with the top cut by a horizontal plane. The walls will be of a constant thickness. The bowl is defined by four parameters: the inner radius of the lower base r_1, the inner radius of the upper base r_2, the thickness d, and the height

h. We will make the upper base minimal (it will be shown that the bowls are turned upside down!), just enough to encircle the cross section of the branches. Thus $r_2=1$. What is the height of the bowl? The Talmud says that there is "a palm with bowl, bulb and flower," and also "there are left three palms with 3 bowls, bulb and flower." Since the bulb itself is a palm long, how could these two sentences be correct? Some commentators explain that in first sentence the bowl and flower are attached to two sides of the bulb, each a palm long, while in the second sentence the bowls are partially inside one another and the bulb is partially inside a bowl. This solution breaks the symmetry of the Menorah. It is also not clear how one can produce the attachment by hammering alone.

There is, however, a very simple and natural solution to the puzzle. The flower contains the bottom half of the bulb and the bowl contains its upper half (see Fig. 1). Hence, the height of the flower and of the bowl is half a palm. In the second sentence, three bowls, bulb, and flower occupy 3x0.5+1+0.5 = 3 palms in successive order! Indeed, in Rambam's drawing, the bowls are turned upside down. One could invert the flowers. However, since the upper flowers form the base of the lamps, their petals should open upwards. This explains Rambam's strange picture. (Remark: According to our solution, the lower bulb is invisible! While studying the journeys of the Children of Israel in the desert, we found a correspondence between the stops and the fixtures of the Menorah. According to this correspondence, the bottom bulb represents the stop at Sukkot, where the Children of Israel were covered for the first time by the clouds [see the end of the article in http://truthofland.co.il/hebrew/chanuka.doc]. In reality, it is possible that there was a small gap between the bowl and the flower, like the gap between the wings of *keruvim*. It is possible to produce such a shape if one starts with a bigger gap, and closes it by applying external pressure.)

We are now left with two parameters: r_1 and d. The only restriction is the given mass of the bowl. We conjecture that the volume of the bowl is exactly the volume of a standard cup, 86.4 cm^3. There are two possibilities: 1) this volume includes the thickness of the walls; 2) it does not. We will consider both options. We now have two equations:

$$(5.1) \quad \begin{aligned} &(a) \quad \pi(r_1^2 + r_1 r_2 + r_2^2)h/3 = V_0 \\ &(b) \quad \pi(r_1 + r_2 + d)dh = V_1 = 17.28 \times 20/19.3, \end{aligned}$$

where $r_2 = 1$, $h = 4$.

In the first case, $V_0 = 86.4 - V_1$; in the second, $V_0 = 86.4$. From the first equation we find r_1, and from the second d. The solution is, in the first case:

$$(5.2)\quad r_1 = 3.44988\ ...,\ d = 0.30000016,$$

and, in the second case:

$$(5.3)\quad r_1 = 3.9583\ ...,\ d = 0.2724234.$$

Notice that in the first case $r_1 \approx 3.4500$, d is extremely close to 0.3, and the exterior diameter of the bowl $2(r_1+d) \approx 7.5$ cm. We will see that the first solution, besides being "almost" rational, also perfectly fits the size of the lamps. (Remark: the "rationality" is with respect to the basic unit of a thumb and is independent of the metric system.) As one can see on the picture, half of the bulb is contained inside the bowl.

6. THE FLOWER

According to the Talmud quotation cited above, the flowers are like the blossoms around the capitals of columns. As we concluded in the last section, their height was half a palm. Since the lower bulb is covered at the base by a flower and at the top by a bowl, we will make the top of the flower identical to the base of the bowl. As for the form of the flower, we will define it as a body of rotation around the central axis. The vertical cross section is shown on Fig. 1. The inner boundary of the section is formed by two identical circular arcs AB and BC, which are tangential at the point of contact, B. The secant ABC is the inner boundary of the vertical cross section of the bowl (with the narrow base at the bottom). Hence, the radius $EA = r_2 = 1$ and $FC = r_1$. The exterior boundary $A_1B_1C_1$ is obtained by shifting the inner boundary to the right by distance d, where the thickness d is the same as that of the bowl. Denote by $r = f(z)$ the curve ABC, $r = g(z)$ the secant ABC, where z is the vertical coordinate and r is the horizontal radius. The volume of the wall of the flower is equal to the volume of the wall of the bowl,

$$(6.1)\quad \pi\int d(2f(z)+d)dz = \pi\int d(2g(z)+d)dz,\ \ 0 \le z \le 4,$$

since the areas bounded by the curve ABC and the secant ABC are the same.

Hence, the mass of the flower is equal to 20 *shekels*. To define the flower uniquely we request the tangent at the contact point B to be vertical. As a result, the radius of the arcs (in the case of r_1 as in (5.2)) is:

$$(6.2)\quad R = ((r_2 - r_1)^2 + h^2)/(r_2 - r_1)/4 = 2.2452.$$

As seen in Fig. 1, the bulb lies in the interior of the flower and the bowl.

7. THE LAMP

The only thing we know about the lamps is their volume: they contained half a *log* or two cups of oil (Mishna Menachot 9, 3). The simplest form of vessel is a cylinder. Interestingly enough, there is a cylinder with ideal measurements, which has almost a required volume—namely, one with diameter and height equal to three thumbs or 6 cm (see Fig. 2). Its volume is 169.646 cm^3 while two cups are 172.8 cm^3. The extra 3.154 cm^3 could be absorbed into the surplus of a height 1.1 mm above the rim, which is about usual for olive oil.

We will assume that the walls and the base of the lamp have the same thickness d (but not necessarily the same as the bowl). The volume of the walls and the base is:

$$(7.1)\quad V = \pi((r+d)^2(h+d) - r^2h), \text{ where } r = 3, h = 6.$$

There are two cases to consider. In the first, the thickness of the base is included in the 18 palms height of the Menorah. In such a case, one should subtract from V the volume of the branch inside the base. If the branch were to enter the base vertically, this volume would be exactly 3d for the dodecagon cross section. The total height of the Menorah with the lamps would be a "nice" number of 150 cm (or 75 thumbs). In the second case, the base is not included. Then, the total height of the Menorah with the lamps would be 150+d. In the first case, we solve the equation:

$$(7.2a)\quad d^3 + d^2(2r+h) + d(r^2 + 2rh - 3/\pi) = 20 \times 17.28/19.3/\pi$$

D = 0.12510 cm, or 1/8 cm up to 1 micron. If the cross section of the branch is a circle of radius 1, the equation becomes:

$$(7.2b)\quad d^3 + d^2(2r+h) + d(r^2 + 2rh - 1) = 20 \times 17.28/19.3/\pi$$

and d = 0.1252 cm, 1 micron more.

In the second case, d = 0.12261, not close to a rational measure. The total height of Menorah is not a "nice" number either.

Now notice how well the lamp fits into the flower. The radius of the arc of the flower is given in (6.2). The upper (inner) arc passes through the point r = 3.45, z = 144, and its center lies at the height z = 142. Hence, the equation of this arc is:

$$(7.3)\quad (z-142)^2+(r-4.47015)^2 = 5.040712.$$

At height z = 144-0.125, r = 3.235, i.e. the distance of 1.1 mm only between the wall of the flower and the base of the lamp. If we were to use solution (5.3), the gap would be much bigger.

8. THE HANDLES FOR THE WICKS

We are left with a total of 20 *shekels*. Our suggestion is that they were used for handles to support the wicks, which were pulled out from the lamps (see Fig. 2). Indeed, Talmud Menachot 98b says: "The seven lamps shall give light in front of the Menorah. This teaches us that they were made to face the middle lamp." This sentence is understood (see Rashi, Bamidbar 8:2) in the following sense. Six wicks extended out of the lamps and slanted toward the central lamp, while the central wick slanted westward, toward the Holy of Holies. However, these wicks require some support. The same Talmud, on p. 88b, says "The lamps in the Temple were made of *segments*." The Talmud further explains: "Like a tray (*tas*) of gold was on it (on a lamp). When he cleaned it, he pushed (the tray) toward its mouth (of the lamp), when he was putting oil into it, he pushed it toward its head (of the lamp)." The word "*tas*" is also mentioned in Talmud Succa 5a: "Mitre is in the shape of *tas* (plate) of gold, two finger-breadths broad and stretching from ear to ear." Clearly, this tray is not the cover of the lamp but an extension of it. These are the *segments* mentioned in the Talmud. The mouth of the lamp is the place where the handle is attached. The head is at the other end of the handle. Normally, the handle is lifted up so that the wick that lies on it will point diagonally toward the central lamp. Therefore, the flame also slants diagonally toward the central lamp. The (normally) lifted end of the handle is called the head of the lamp. When he (the priest) was cleaning the lamp, he pushed the handle down to the level of the mouth and a little below. The wick then pumped the leftovers of the oil from the lamp to a vessel. After filling the lamp with fresh oil, the handle was lifted up toward the position of the head, and accommodated a new wick.

In designing the form of the handles, we will use the standard measures, a palm and a thumb. Let us remember that the distance between the centers of the lamps is two palms. Since the lamps are 6.25 cm wide, the distance between the lamps is 9.75 cm, a little more than a palm. We will assume that the handles were a palm long. In order to accommodate the wick we will assume that they were a half cane. The diameter of the cane will be a thumb, like the branches of the Menorah. If the thickness of the cane is d, then:

$$(8.1)\quad 8\pi d = 20\text{x}17.28/19.3/7 = 2.55810\ldots,\ d = 1.018\text{ mm}$$

We will round it off to a rational measure of thumb/20 = 1mm. (Remark: This is apparently the standard minimal measure called "the thickness of a golden *denarius*.") Notice that the left-hand side of (8.1) gives the volume of a cane with radius 1 measured from the center of its thickness. If, for example, we measure the radius r=1 from 1/3 of the thickness, then the volume of the handle with d = 0.1 will be:

$$(8.2)\quad 4\pi((1+2d/3)^2-(1-d/3)^2)=2.555,$$

almost the requested one. The exact form of attachment of the handle to the lamp requires additional deliberation. In Fig. 2 we presented a simplified picture of the attachment using a flat handle.

9. THE MENORAH OF RAMBAM

In Section 3 we made an approximate calculation of the weight of the Menorah of Rambam, namely, the Menorah with straight branches. This calculation does not take into account the mutual intersections of branches and legs. Since these intersections will decrease the weight of the Menorah, we can balance it by increasing the thickness of the branches. Namely, we will assume that the branches have circular cross section of radius $r_0 = 1$. The intersection of branches with the bulb depends on the size of the bulb. The corresponding semi-axes a and b have been calculated in (4.7).

The equation of a side branch is:

$$(9.1)\quad (x-(z-z_i)/\tan\alpha_i)^2/a_i^2+y^2/r_0^2=1, \text{ where } a_i=r_0/\sin\alpha_i,$$

$$\alpha_i=\operatorname{atan}((144-z_i)/x_i),$$

$$x_1=48,\ x_2=32,\ x_3=16,\ z_1=72,\ z_2=88,\ z_3=104.$$

Notice that we assumed that the axis of the branches intersects the axis of the stem at the center of the bulbs.

The equation of a bulb is:

$$(9.2)\quad x^2/a^2+y^2/a^2+(z-z_i)^2/b^2=1,$$

with a and b given in (4.7). For given z, the intersection of these two bodies is the intersection of two ellipses. One can easily write the formula for the area of the branch outside the bulb. We integrated this area with respect to z numerically. The volumes of three pairs of branches up to the very top z = 144 are, correspondingly:

$$(9.3)\quad V_i = 528.6045,\ 388.4730 \text{ and } 249.9793\ \text{cm}^3,\ i = 1,2,3.$$

It turns out that the two upper branches (i=3) also intersect the stem above the bulb (see Fig. 3). The corresponding volume is:

(9.4) $\Delta V_3 = 0.401 \text{ cm}^3$

for each branch. In addition, we should take into account the intersection of the branches with the base of the lamps. Its volume is:

(9.5) $\pi r_0^2 / \sin\alpha_i \cdot d$, α_i is in (9.1), d as in (7.2b).

However, the part $\pi r_0^2 d$ was already subtracted from the volume of the lamp in (7.2b). Thus, we should subtract from the volume of the branches the difference:

(9.6) $\Delta V1_i = (\pi r_0^2 / \sin\alpha_i - \pi) \cdot d \approx 0.08, \quad 0.06, 0.03$, d as in (7.2b).

Most difficult is the calculation of the volume of the legs. We have assumed, in Section 3, that the center of the base of the legs, like the top of the upper branches, is two palms away from the axis of the Menorah. If we place this center on the x axis, the equation of the leg becomes:

(9.7) $(x - 16 + z / \tan\alpha_4)^2 / a_4^2 + y^2 / r_0^2 = 1, a_4 = r_0 / \sin\alpha_4.$

The other two legs are obtained from the above, by rotation around the axis of the Menorah with angles 120° and 240°. The leg touches the stem at:

(9.8) $z1 = (16 - r_0 - r_0 / \sin\alpha_4) \cdot \tan\alpha_4$

and the axis of the Menorah at:

(9.9) $z2 = (16 - r_0 / \sin\alpha_4) \cdot \tan\alpha_4$

The exterior side of the leg touches the stem at:

(9.10) $z3 = (16 - r_0 + r_0 / \sin\alpha_4) \cdot \tan\alpha_4$

This is the top of the leg (see Fig. 4). Recall that, according to Talmud Menachot 28b: "legs and the flower three palms." Since the flower is above the legs and is a half-palm high, the legs are 2.5 palms or 20 cm high. Thus, we obtain the equation for the angle α_4:

(9.11) $(16 - r_0 + r_0 / \sin\alpha_4) \cdot \tan\alpha_4 = 20, \alpha_4 \approx 50.84°$

We will first assume that the stem extends down to the level $z = z1$. The volume of the legs should be calculated from z1 up to z3 outside the stem. Care should be taken with regard to the mutual intersection of the legs. All these computations were carried out analytically, and then compared with numerical integration. The resulting integral is:

(9.12) I_{13}=21.5472 cm^3

(the mutual intersection of the legs is 3 x 0.0438 cm^3). The volume of three legs from the bottom up to z1 is:

(9.13) $I_1 = 204.6116\ cm^3$.

The volume of the stem from z1 up to z = 144 is:

(9.14) $I_{cent} = 399.5068\ cm^3$

(notice that there is no correction for the central stem as in (9.6)). The final volume of the body of the Menorah is:

(9.15) $V_{bd} = I_1+I_{13}+I_{cent}+\Sigma(V_i-2\Delta Vl_i)-\Delta V_3 = 1791.983\ cm^3$

and the corresponding weight 2001.463 *shekel*s. Notice that part of the stem from z1 up to z2 is unnecessary. The three legs meet together at the axis of the Menorah at z = z2 (see Fig. 4), and surround a cone of height z2-z1 and base πr_0^2. The volume of this cone is 1.2857 cm^3 and its weight is 1.436 *shekel*s. If we remove this cone, the weight of the body of the Menorah will be 2000.027 *shekel*s. Practically speaking, this is identical to 2000 shekels. If we define the thickness d of lamps to satisfy (7.2a) instead of (7.2b), then the weight of the body of the Menorah will be 1999.89 *shekel*s. Note that we had no free parameters to play with. The only choice was to cut out the cone surrounded by the legs. In this way, the legs extend from a dome-like base, as in the picture drawn by Rambam. The only problem with this Menorah is the adjustment of the lamps to the upper flowers, or the flowers to the upper bulbs. The Menorah is shown in Fig. 3. We placed the center of the base of all 7 upper bulbs at the same level, z = 132. Notice that, due to different slopes of the branches, the flowers are at different levels. The axes of the lamps pass through the centers of the top of the branches. The two flowers at the top of the two upper branches almost touch the lamps, but do not intersect them.

10. CIRCULAR BRANCHES

The Menorah described above accords with the opinion stated in Talmud Menachot 88b, that the lamps were included in the 3000 shekels. This is also the ruling of Rambam in *Halachot Beit Habechira* 3:6. However, there is another opinion stated in the Talmud, that the lamps were separate from the Menorah and were not included in the 3000 *shekel*s. With fixtures defined as in previous sections, we are left with an extra 160 *shekels* of gold for the body. Hence, the branches could be made longer. It might even be possible that the branches were circular.

To this end, we will make the lower part of each branch a quarter of a circle:

$$(10.1)\quad x^2 + (z-120)^2 = R_i^2,\ R_i = 48, 32, 16,$$

and the upper part vertical. Note that the origin of the branches 9, 7 and 5 palms below the top of the Menorah and their span of 6, 4, and 2 palms respectively, implies that the vertical part of the branch is the same for all branches. It is exactly three palms, as the space occupied by three bowls, a bulb and a flower! We will make the legs too in the same form—with the upper part a quarter of a circle:

$$(10.2)\quad x^2 + (z-3)^2 = R_4^2,\ R_4 = 16,$$

and the lower part vertical. Since the central line of the leg originates at $z = 19$, and since its span is 16 cm, its vertical part will be 3 cm. Now, the Menorah stands firmly on its legs and the lamps are vertical and fit the branches. The Menorah is shown in Fig. 5. The legs are shown schematically, as if they were in the plane of the Menorah. It turns out that, in order to have the correct weight, the cross section of the branches should be a dodecagon of area 3 cm^2, as described in Section 3. At each point of a circle defined by (10.1) or (10.2), we construct the dodecagon with its center at this point, and perpendicular to the tangent to the circle (the normal to the circle will be one of the axes of the dodecagon). The branch is defined as the set of such dodecagons. It is easy to see that the volume of the branch is equal to the length of the circle multiplied by the area of the cross section 3 cm^2. From this volume, we should subtract the part of the branch that is common with the bulb. In the first approximation, we consider the side branch as a horizontal cylinder of radius $r = \sqrt{(3/\pi)}$. The corresponding volume is given by the integral:

$$(10.3)\quad I = \int\int a(1 - y^2/a^2 - z^2/b^2)^{1/2}\,dzyd,\ \ y^2 + z^2 \le r^2,$$

where a and b are defined in (4.5). This integral is transformed into polar coordinates, integrated analytically with respect to the radius and numerically with respect to the polar angle. Its value is 3.9807 cm^3. In the second approximation, we add to it the difference between the actual length of the circle (10.1) inside the bulb and semi-axis a, multiplied by the area of cross section 3. To the leading order in a_i, it is:

$$(10.4)\quad \Delta I_i = 2a^3 a_i^2,\ \ a_i = 1/(2R_i),\ \ R_i \text{ as in } (10.1).$$

The legs require special attention. Recall that, according to the Talmud: "legs and the flower three palms." Since the flower is above the legs and is half a palm high, the legs are 2.5 palms or 20 cm high! Since they are 2 cm thick, the central line of the leg starts at the height $z_4 = 19$.

From the above, one should subtract the common volume of the leg and the stem. Both are considered cylinders of radius r, the leg being horizontal. The corresponding volume is given by the integral:

$$(10.5)\ \ I_4 = \int\int (r^2 - y^2)^{1/2} dzdy,\ \ y^2 + z^2 \le r^2.$$

Its value is found analytically, $I_4 = 8r^3/3$. As in (10.4), one should add:

$$(10.6)\ \ \Delta I_4 = 2r^3 a_4^2,\ \ a_4 = 1/(2R_4).$$

There is, however, a mutual intersection of the legs outside the stem. Again, for simplicity, we assume that the legs are horizontal cylinders of radius r with a 120° angle between their axes. If we cut them by a horizontal plane $z = z_4 + h$, we obtain three semi-infinite strips of width $2\sqrt{(r^2-h^2)}$. Their intersection can be split into three cusps, as shown in Fig. 6. The area of cusp ABC is approximated by the area of triangle EFC. Its height is $\sqrt{(4/3(r^2-h^2))}-r$ and the top angle C =120°. Thus, the said volume is given by the integral:

$$\begin{aligned}(10.7)\ \ \Delta V &= 3\tan 60^\circ \int (\sqrt{(4/3(r^2 - h^2))} - r)^2 dh,\ \ -r/\sqrt{3} \le h \le r/\sqrt{3} \\ &= 12r^3(7/6-2/27-2(\pi/4+\sin(2\alpha)/4-\alpha/2)),\ \alpha = \arccos(1/\sqrt{3}) \\ &= 0.064\ cm^3.\end{aligned}$$

The main stem starts at the height of 18 palms = 144 cm and descends to the point where the legs completely leave the stem. If the legs were horizontal cylinders of radius r, this height would be 19-r. Since the legs are circular, the height becomes $19\text{-}r\text{-}a_4r^2$. The length of the stem is:

$$(10.8)\ \ l_0 = 125+r+a_4r^2.$$

The total volume of the body of the Menorah is:

$$(10.9)\ \ V_{total} = \pi r^2(l_0+2l_1+2l_2+2l_3+3l_4)-6I-3I_4-2(\Delta I_1+\Delta I_2+\Delta I_3)-3\Delta I_4-\Delta V.$$

The length of the branches and the legs is calculated trivially. We obtained the total volume of the body of the Menorah 1936.561 cm^3 and its weight 2162.9416 *shekels*. Thus, we have an extra 2.9416 *shekels*. However, we forgot to make one adjustment. The tops of the branches have a common volume with the base of the lamps. This volume is 3d for a branch, 21d = 2.625 cm^3 total (see 7.2a). Since the lamps together with their bases are not included in the 3000 *shekels*, the above volume should be not included in the volume of the body of the Menorah. The weight now diminishes by 2.625 x 19.3/17.28 = 2.932 *shekels*. We are left with a negligible excess of 0.01 *shekel*. This is indeed a miracle! We had no parameters to play with. Thus, there is controversy between the straight Menorah of Rambam, which includes the lamps in the 3000 *shekels*, and an ideal circular Menorah, which does not include the lamps.

11. PARABOLIC MENORAH

The Menorah portrayed on a Hasmonean coin and on the wall of a house from the Second Temple period, has parabolic rather than straight or circular branches. Let us check whether a Menorah of such form would have the requested weight. With x being the horizontal coordinate measured from the axis of the Menorah, and z the vertical coordinate measured from the base of the Menorah, the equation of the central line of a side branch is:

$$(11.1)\quad z = z_i + a_i x^2, \text{ where } z_1 = 72,\ z_2 = 88,\ z_3 = 104.$$

The coefficients a_i are found from conditions

$$(11.2)\quad z_i + a_i x_i^2 = 144, \text{ where } x_1 = 48,\ x_2 = 32,\ x_3 = 16.$$

The length of a parabola

$$(11.3)\quad l_i = \int \sqrt{(1+4a_i^2x^2)}dx,\ 0 \le x \le x_i$$

is given by the formula

$$(11.4)\quad l_i = [t\sqrt{(1+t^2)} + \ln(t+\sqrt{(1+t^2)})]/4a_i, \text{ where } t = 2a_ix_i.$$

Since the branches are longer than those of Rambam's Menorah, we will assume that the cross section of the branches is a dodecagon, as in Section 10. Given the central line, the shape of a branch is defined as in Section 10. Thus, the volume of the branch is equal to the length of the parabola multiplied by the area of cross section 3 cm². From this volume we should subtract the part of the branch that is common with the bulb. This is done as in (10.3) and (10.4); with a_i as in (11.2).

The legs are also parabolic. The central line is a parabola with the top at z_4=19 and a span of R_4=16 at the base. This defines the coefficient a_4 of the parabola. The intersections of the legs are the same as in (10.5)-(10.7). The length of the stem is the same as in (10.8). It turns out that the total volume of the Menorah is 1793.38 cm³, and the corresponding weight in *shekels* is 2003.023. We have an extra 3 *shekels*! If, however, we make the span of the bottom branches 48-0.75 cm instead of 48 cm, then the weight will be 1999.987 *shekels*! But what is the justification for such a span? Recall that there are flowers at the top of the branches, of exterior diameter 7.5 cm. The total width of the Menorah will be:

$$(11.5)\quad 2(48-0.75) + 7.5 = 102 \text{ cm}.$$

This number is exactly two cubits of the Temple!

For further discussion on the parabolic Menorah and additional possible forms of the Menorah, see the article at http://truthofland.co.il/menora_form/form.htm .

12. STRENGTH OF THE MENORAH

The last test is whether the Menorah can sustain its own weight. The most problematic case is the one with circular branches. Of all branches, the lowest one produces the largest angular momentum at the point where it joins the bulb (see Fig. 5). Recall that the branch consists of a quarter of a circle with radius R = 48 cm and vertical part of length l = 24 cm. The cross section of the branch has an area of S = 3 cm^2. The angular momentum of the weight of the branch is:

$$(12.1)\quad M = (R^2 + Rl)Sdg\,,$$

where d = 19.3 g/cm^3 is the density of gold and g = 980 cm/sec^2, the Earth's gravity. In addition, we have the momentum of the fixtures: 3 bowls, bulb, flower and lamp with handle and oil. Their total weight is about 130 *shekels* and their angular momentum is about 130 x 17.28gR. This momentum is balanced by the momentum of stress at the joint with the bulb. We may assume that the cross section of the joint with a vertical plain is a circle $x^2+z^2 \le r^2 = 3/\pi$. To a first approximation, the stress σ is a linear function of the vertical coordinate z, $\sigma = \sigma_0 z/r$. The momentum is:

$$(12.2)\quad \sigma_0 / r \int z^2 dxdz = \sigma_0 \pi r^3 / 4.$$

We obtain

$$(12.3)\quad \sigma_0 \approx 44 \times 10^7 \,\mathrm{din/cm^2} = 44\ \mathrm{MPa}.$$

The yield point of annealed 24 carat gold is 70 MPa. If the stress does not exceed this value, the material will return to its origin shape as the stress is removed.

Another critical point is the joint of the legs and the main stem. The force acting on the base of the leg is 1000 *shekels*. One should add to this the weight of lamps, handles, and oil since they are not included in the 3000 *shekels*. The total force is about 1076 x 17.28g and the moment arm is 16 cm. Its angular momentum is about 97 percent of the angular momentum for the lower branch. Thus the legs will also not deform. Notice that, for cold worked gold, the yield point is higher (144 MPa for 20 percent cold work, 205 MPa for 60 percent cold work). Indeed, the Menorah was manufactured by hammering, and hence was much stronger. Yet, the Talmud in Menachot 29a describes how the Menorah of the Temple had an excess of a *dinar* of gold, and was put into a furnace 80 times until the weight was right. After heating in a furnace, the gold becomes annealed. Hence, the strength of the Menorah should be calculated for the annealed gold.

13. MANUFACTURING THE MENORAH

According to *Baraita Melechet Hamishkan*, the Menorah, in its entirety, was hammered out of a single piece of gold. No casting was used. *Miksha ahat*, the instruction written in the Torah, means "one piece," but the word *miksha* also means "beaten work." This poses a formidable difficulty in manufacturing the Menorah. However, the form of the original piece of gold is not specified. So, one could suggest casting the Menorah in a shape close to the desired form, and then finishing it with few strokes of a hammer. Yet, this does not seem to be the Torah's intention. As for the form of the original piece, I suggest a ball—the simplest body of given volume. Perhaps, this is what is hinted at in Zechariah 4:2: "a candlestick all of gold, with a ball (Hebrew: *hagula*) on the top of it." Since we know the weight of all the parts of the Menorah, we could solve the problem if we could make every part of the Menorah while separating it from the rest. The separation can be implemented by the use of bands, if we can find a way to prevent a flow of gold through the band under pressure. It is also permissible to heat the Menorah, or any part of it, to reduce the rigidity. Many special tools need to be designed, and many small-scale experiments need to be carried out before the necessary experience will be gained.

14. CONCLUSION

In this article we have reconstructed three different kinds of Menorah: straight, round, and parabolic. In the full article on the Web, two more forms are presented. Which is the correct one? It is stated in I Kings 7:49 that King Solomon made ten menorahs in addition to the Menorah of Moses. It is possible that they were of eleven different shapes and represented eleven different solutions to the problem. Which of them was the Menorah of Moses? My guess is that this was the straight Menorah of Rambam. It has the simplest form, fits the ruling of Rambam, and has the exact weight without any adjustment. The wicks converging at the stem form an upper triangle of fire that complimented the lower diverging triangle of golden branches.

15. APPENDIX—THE SUMMARY OF ASSUMPTIONS

For the reader's convenience, we have collected here the different assumptions made in the course of the paper:

1) The *shekel* of the Torah is related to the unit of volume, the cup, or the quarter *lug* full of distilled water at 4°C, as 1:5.
2) The density of hammered pure gold at room temperature is 19.3 gr/cm^3.
3) The width of the Menorah is 12 palms, like the length of the table. This width is divided into six equal parts—the distances between the tops of neighboring branches and the stem. More precisely: the distance between the axis of the Menorah and the centers of the top of the branches is 2, 4 and 6 palms, respectively.
4) The center of the base of the legs is two palms away from the axis of the Menorah, as are the centers of the top of the inner branches.
5) The legs are 2.5 palms high. This follows from the fact stated in Menachot 28b that the legs and the flower are three palms and, as inferred by us, that the height of the flower is half a palm.
6) The branches and the stem are a thumb thick, where a thumb is 1/24 of a cubit.
7) The bulb is an ellipsoid of rotation as exhibited in Fig.1. Its height is a palm, as stated in Menachot 28b.
8) The bowl is a cone half a palm high, as shown in Fig.1. Its volume, including the walls, is one cup.
9) The flower has the form shown in Fig. 1 and is half a palm high. Notice that the bowl and the flower do not affect the volume of the body of the Menorah; however, the height of the flower affects the height of the legs, as in 5).
10) The lamps have cylindrical form as shown in Fig.2. Their inner diameter and height are three thumbs, so that their volume is almost exactly two cups—as stated in Mishna Menachot 9, 3. The walls and the base of the lamps have a uniform thickness. The thickness of the base is included in the height of 18 palms of the Menorah.
11) The axes of the branches intersect the axis of the stem at the heights 9, 11 and 13 palms from the base. As a result, the lower branches separate from the stem exactly at the middle of the height of the Menorah. In the case of the round Menorah, the vertical segments of the branches are 3 palms long and fit the space occupied by three bowls, bulb, and a flower. However, according to Menachot 28b, the origin of the lower branches with the bulb occupy the 9th palm, the middle ones occupy the 11th palm and the upper ones the 13th

palm. Thus, if we follow the exact wording of the Talmud, the points of intersection of the axes should be 8.5, 10.5 and 12.5 palms from the base. As a result, the branches would become longer and the weight of the Menorah would increase by 74 *shekels* in the case of the straight Menorah, and by 80 *shekels* in the case of the round Menorah. The Menorah would also lose its "harmony": the lower branches would separate from the stem not at the middle of the height of the Menorah but half a palm below it. In the case of the round Menorah, the vertical segments of the branches would become 3.5 palms long and would not fit the space occupied by three bowls, bulb, and a flower. Hence, our interpretation of the text in Menachot 28b is as follows. Since all measures of the Menorah in the Talmud are given in integer numbers of palms, it counts 2.5 palms between the lower bulb and the next one as two palms.

12) For the straight Menorah, the branches and the legs are straight. Their cross section and that of the stem is a circle of diameter one thumb.

13) For the round Menorah, the branches and the legs are initially quarter circles and then vertical. Their cross section is a dodecagon inscribed in the circle of diameter one thumb.

14) All 49 fixtures of the Menorah: the 11 bulbs, 22 bowls, 9 flowers and 7 lamps have the same weight – 20 *shekels* of 17.28 gr, or the *mane* of the Second Temple. The lamps have handles attached to them. The total weight of the seven handles is also 20 *shekels*. The intersection of the stem with the bulb is not included in the weight of the bulb. Likewise, the intersection of the stem with the base of the lamp is not included in the weight of the lamp.

15) For the round Menorah, the lamps with their base and handles are not included in the 3000 *shekels* weight of the Menorah. This is the second opinion in Menachot 88b.

16) For the straight Menorah, the cone with the base at level z_1 and the vertex at z_2 are cut out from the stem as shown in Fig. 4.

17) For the parabolic Menorah, the axes of the branches and the legs are parabolas originating at the axis of the Menorah. Their cross section is the same as that of the round Menorah. The rest of the parameters are the same as for the straight Menorah.

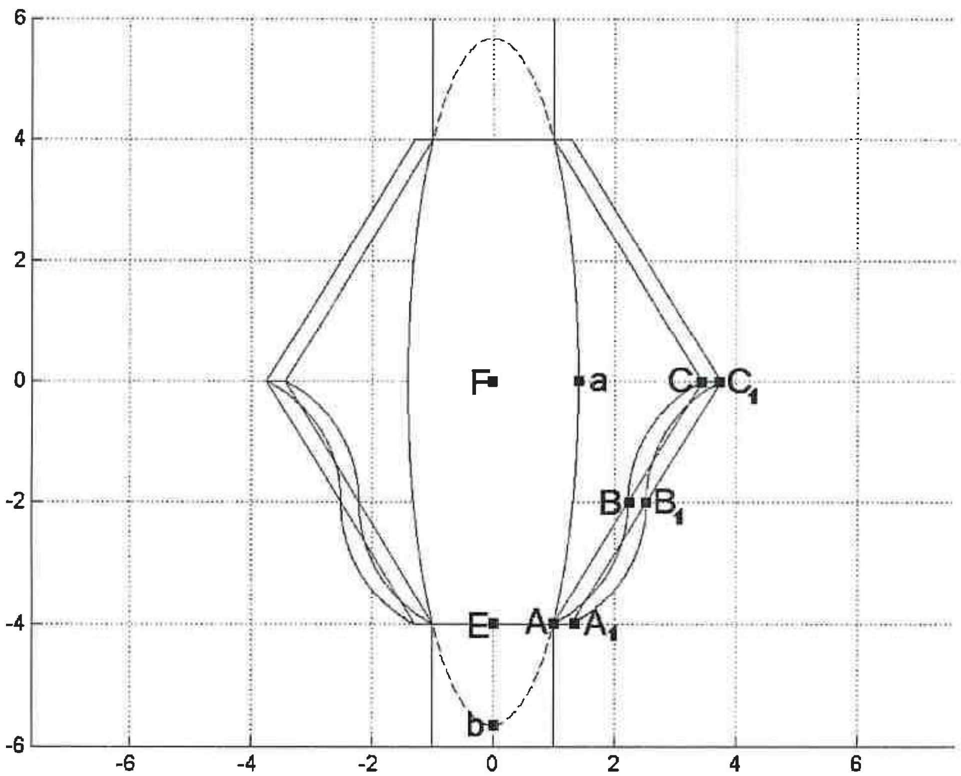

Fig. 1 Bulb, flower and bowl

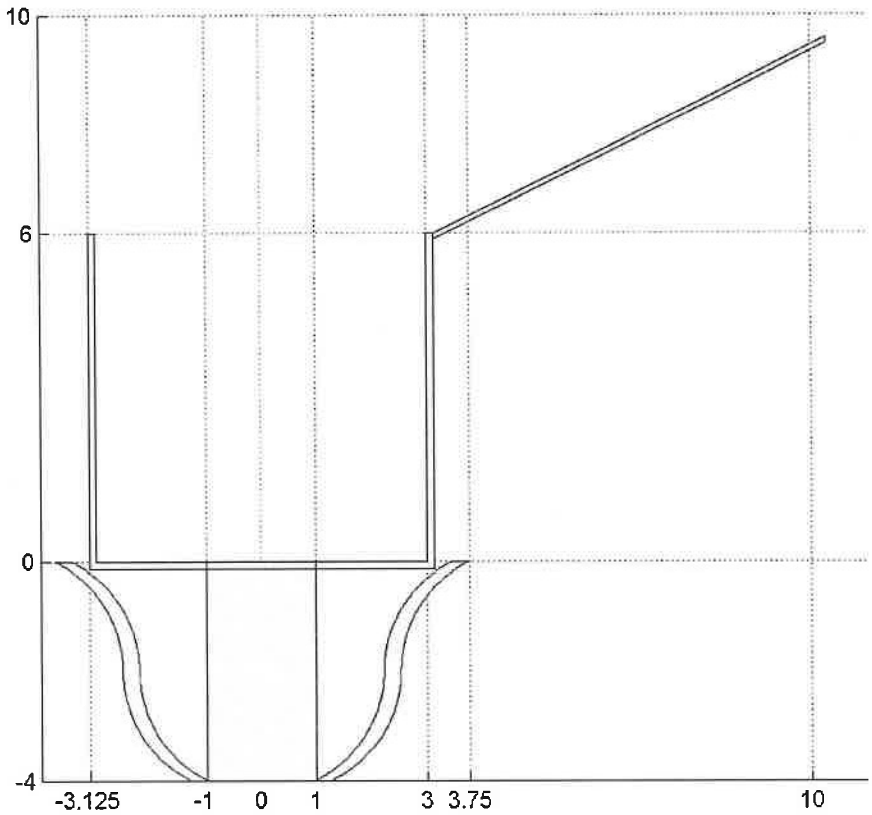

Fig. 2 The lamp

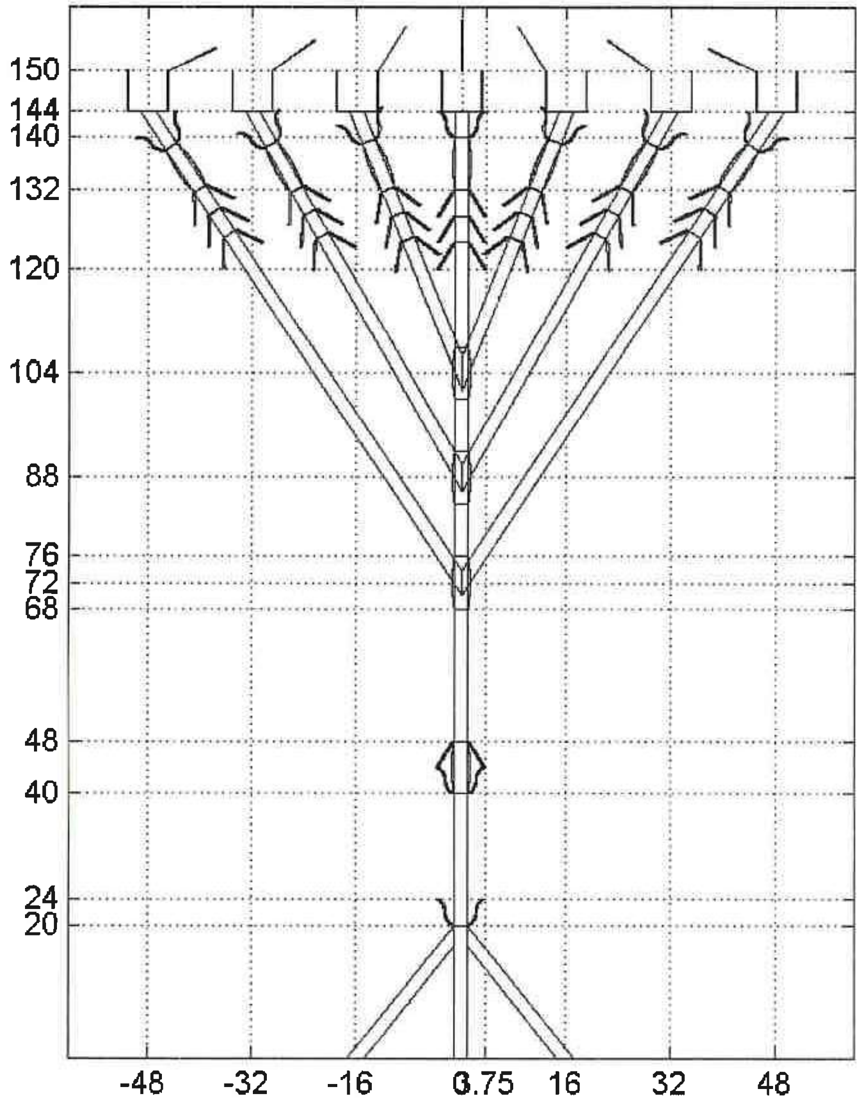

Fig. 3 The straight Menorah

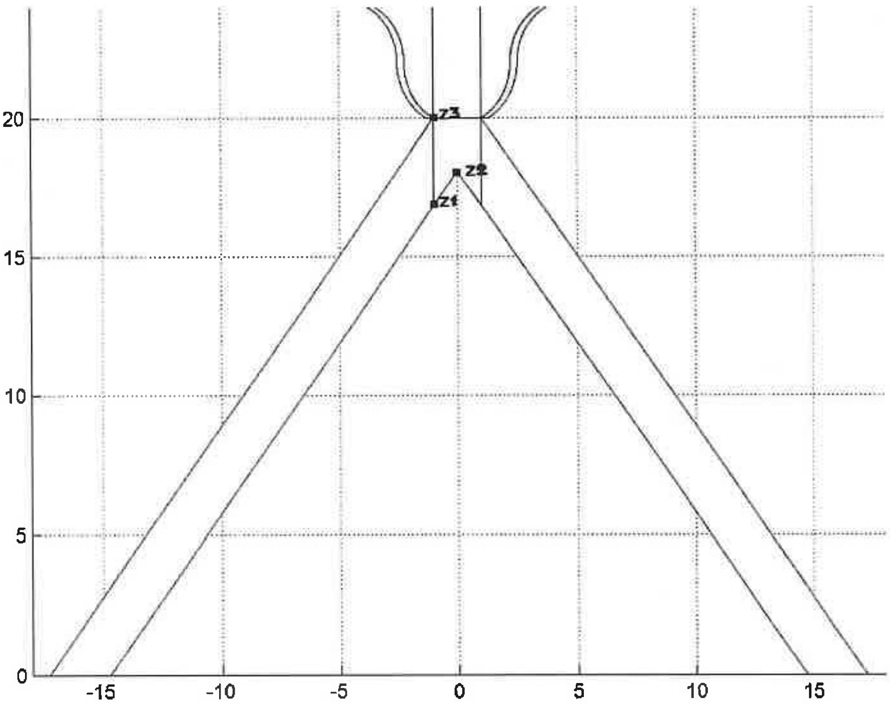

Fig. 4 The legs of the straight Menorah

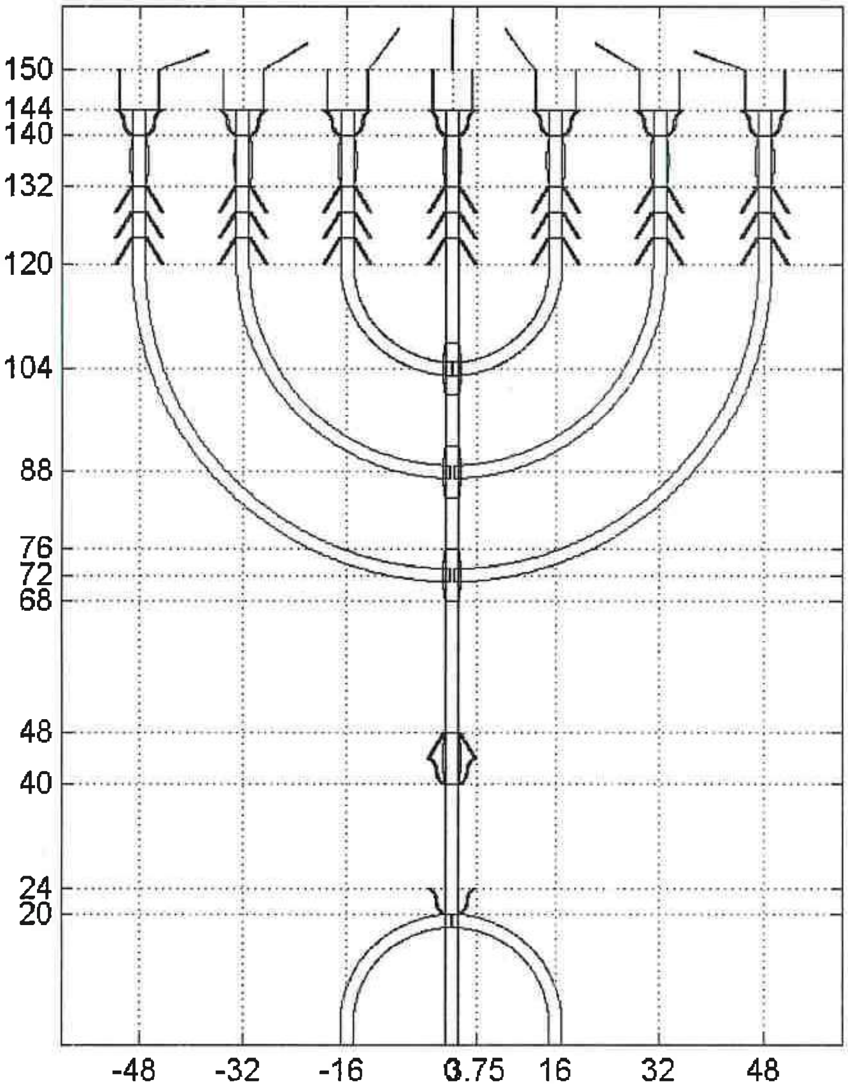

Fig. 5 The round Menorah

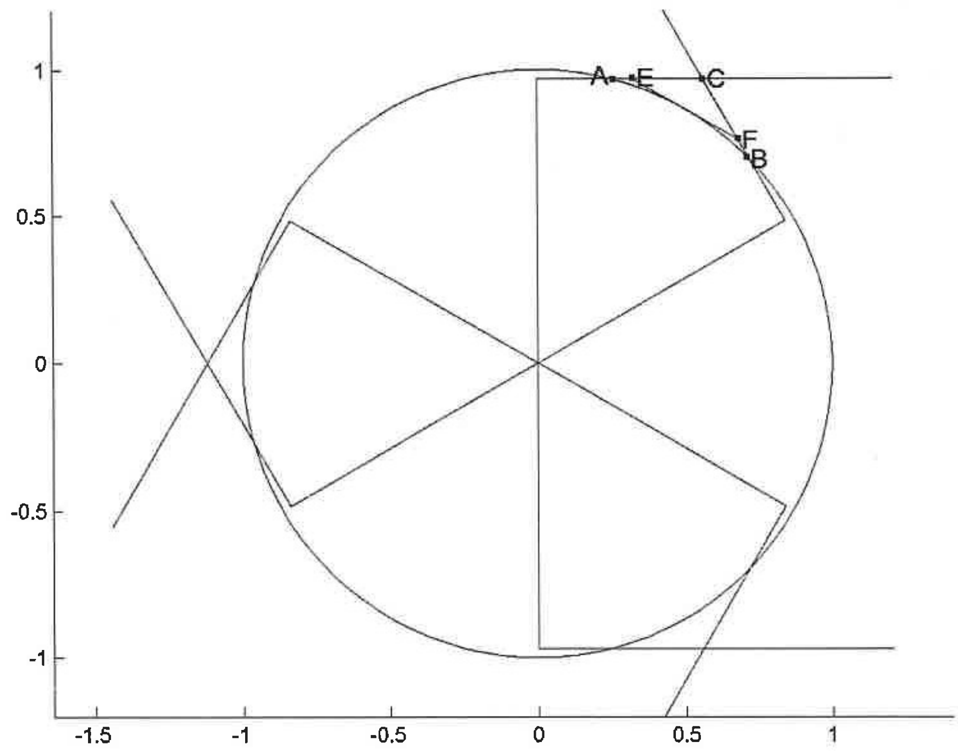

Fig. 6 Intersection of the legs