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THE APPLICABILITY OF INFINITESIMALS AND THE FOUNDATIONS OF THE CALCULUS TO THE LAW OF NULLIFICATION

The calculus could be summarized as the formalization of a method for dealing with the infinitely small. This paper traces the history of man's pursuit to gain insight into the infinitely small on both a theoretical and a practical level. It demonstrates the origins of this pursuit in ancient philosophy, its important role in mathematics and the calculus, and its fundamental significance in modern physics. The paper provides new insight into the relevance of the infinitely small to halakhic issues by exhibiting its applicability to the Law of Nullification.

1. INTRODUCTION

The most significant mathematical creation of the 17th century, and the one that proved the most fruitful for the modern development of mathematics and science, is the calculus (Kline, 1985, p. 365). The heart of the calculus lies in its formalization of a method to study the infinitely small. This method allows for the computation of instantaneous velocity (derivatives) and the area under a curve (integrals). The enigma of the infinitesimal, however, is not specific to the calculus or even to mathematics. Throughout the centuries, man has frequently encountered the same dilemma, trying to bridge the gap between the quantifiable and the infinitely small. The basic conundrum that presents itself to any thinker is whether matter is infinitely divisible or if it reaches a point of indivisibility.[1] This mathematical problem, which was successfully handled by Newton and Leibnitz, has plagued experts in multiple disciplines. It was tackled by the Greek philosophers Zeno and Aristotle, the modern physicists Einstein and Hawking, and the Talmudic scholars *Rabbeinu Tam* and the *Riva*. Despite the generality of the problem, each discipline approaches it in its own unique way.

The basic dilemma is as follows: The mind naturally assumes that just as matter can be divided once, it can be divided indefinitely. There is no reason to assume

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1 This dilemma could be described as whether one should view matter as being continuous or discrete (Bell, 1995, pp. 55-57).

that there is a limit to this divisibility. But, if everything can be divided infinitely, what are the building blocks of matter? How can any substance emerge? It seems as if there must be some "indivisibles" from which all else is composed. Additionally, it becomes practically difficult for a scientist in any discipline to deal with ever-shrinking quantities. The assumption of atoms, infinitesimals, or "indivisibles" simplifies one's scientific pursuits. This being the case, both the assumption of the indivisibility of matter, and that of infinite divisibility of matter are fraught with their own difficulties. These are the difficulties that many great thinkers have encountered. This paper traces the history of this dilemma, and elucidates its emergence in multiple fields of study. It demonstrates that Torah scholars were well aware of the difficulties inherent in the analysis of the infinitely small by citing explicit references to its treatment in both halakhic and philosophical Jewish texts. New insight is provided into the expression of this dilemma in the context of Halakhah. Its applicability to a debate between Rabbeinu Tam and the *Riva* regarding a specific instance of the Law of Nullification (*bitul*) is fully elucidated.

Section 2 of this paper provides a brief account of the history of the infinitely small. It discusses the origin of this issue in the sphere of philosophy, traces its development to mathematics and the development of the calculus, and demonstrates its unique applicability to the study of physics. Section 3 is devoted to surveying Jewish literature, both halakhic and philosophical, which discuss the infinitely small. Section 4 provides new insight into the relevance of the infinitely small in the setting of Halakhah. Section 5 consists of some concluding remarks.

2. A BRIEF ACCOUNT OF THE LARGE HISTORY OF THE INFINITELY SMALL

2.1 The Philosophical Perspective

Historically, the infinitely small first vexed the ancient Greek philosophers. Anaxagorus (c. 500 BCE) claimed, "In the small there is no least, but always a lesser" (Wheelwright, 1960, p. 161). He thereby advanced the doctrine of unlimited divisibility of matter, a position later adopted by Aristotle (Aristotle, 1984, pp. 404-407 and 1711-14). In the opposite camp were the atomists, who maintained that all matter was composed of weightless atoms that could not be further subdivided (Wheelwright, 1960, pp. 175-99).[2] This philosophical dispute was not resolved by the ancient Greeks, but has puzzled even the most modern philosophers.

2 In fact, the word "atomos" in Greek means "indivisible."

Immanuel Kant suggests a resolution to the apparent contradiction between the two seemingly innate notions in man; that of the infinite divisibility of matter and that of the existence of basic building blocks of matter (Kant, 1977, pp. 74-77).[3]

2.2 The Mathematical Perspective

In his formalization of mathematical methods to study the universe, man came face to face with this quandary time and again. Assuming Aristotle's proposition of infinite divisibility of magnitudes, Eudoxus (c. 370 BCE) devised a method for solving mathematical problems (Boyer and Merzbach, 1991, pp. 88-92).[4] Archimedes, representing the opposing school of thought, was the first mathematician to adopt the atomist position and make use of infinitesimals in deriving mathematical results (Eves, 1990, pp. 383-85).

Although the methods of Archimedes proved to be mathematically fruitful, little progress was made in these areas until centuries later. Among the most noted Europeans in the 16th and 17th centuries to make progress in the application of infinitesimals to integration-like problems were Kepler, Cavalieri, Torricelli, Fermat, Pascal, and Barrow. Due to the lack of a solid theoretical basis for infinitesimals, their methods met with much criticism from other mathematicians. However, the forward momentum was too strong, and many of the principles of the integral and differential calculus were discovered under the assumption of the existence of infinitesimals. Despite this progress, there were two major aspects of the calculus that were missing. Firstly, there was a need for a systematic set of rules in order to make these subjects more concrete. Additionally, a rigorous conceptual understanding of the fundamentals of these subjects was still lacking (Eves, 1990, pp. 385-97).

The first of these developments was provided by "the founders of the calculus," Isaac Newton and Gottfried Wilhelm Leibnitz. In the 1660s, Newton developed the major ideas upon which the calculus is based, and devised methods for performing differentiation and integration. These methods made use of infinitesimals, or *moments,* as Newton called them. In the 1670s and '80s, Leibnitz independently developed the same ideas, although described in different terms. He introduced the symbols dx and dy of differentiation, and the symbol $\int$ of integration, used by modern-day mathematicians in the study of the calculus (Cajori, 1929,

3 For a discussion of various philosophers' positions regarding the infinitely small and its applicability to the calculus, see Laugwitz (1997b).

4 Eudoxus' Method of Exhaustion was used to prove results involving areas and volumes of curvilinear figures.

pp. 201-206). The work of Newton and Leibnitz concretized methods that their predecessors utilized only informally (Hellemans and Bunch, 1991). Despite these powerful methods, the fundamental basis of their work was still questionable. Both Newton and Leibnitz followed the ancient philosophical position of the atomists, assuming the existence of an infinitely small quantity. The philosopher George Berkeley attacked their ideas, and argued that they were treating these infinitesimals as zero and nonzero at the same time.[5] Major disagreement once again arose from man's difficulty in grasping the infinitely small. The underlying dilemma that had divided the ancient philosophers resurfaced to split the European mathematicians.

Despite the sharp criticisms of its conceptual basis, the great mathematicians of Europe accepted the new discipline of the calculus and worked to develop it further (Berlinski, 1995, p. 114). It was not until the 19th century that mathematicians began to realize that they needed to gain a greater understanding of the theory on which the calculus is based (Eves, 1990, pp. 565-67). There were a few unsuccessful attempts to make the calculus more rigorous, until the task was successfully accomplished by the French mathematician Augustin-Louis Cauchy, in 1821. He developed a theory of limits,[6] which allows one to deal with ever-shrinking quantities by considering what happens as they get closer and closer to zero, without positing a mysterious quantity called an infinitesimal (Berlinski, 1995, pp. 118-20). Cauchy thus aligned the calculus with Aristotle's doctrine of the unlimited divisibility of matter, as opposed to the atomists' notion of its indivisibility. Since Cauchy, the foundations of the calculus have been understood on a more in-depth level. However, the basic assumption of the infinite divisibility of matter has remained the cornerstone of standard analysis (Eves, 1990, pp. 565-69).[7]

As mathematics progresses further and further, man continues to encounter the same underlying quandary that divided the Greek philosophers. The issue of how to regard quantity as one approaches zero continually perplexes man. He is torn between assuming the existence of an infinitely small quantity that cannot be divided, and the supposition of the infinite divisibility of quantities. Both

5 For a thorough analysis of Berkeley's criticism and its merits, see Grattan-Guinness (1969).

6 For insight into Cauchy's development of the rigorous foundations of the calculus, see Grabiner (1981, 1983).

7 More recently, however, the concept of the infinitesimal has reappeared on more solid grounds. Nonstandard analysis and synthetic differential geometry (1960s and 70s) have reintroduced the atomistic approach into the sphere of mathematics, after it seemed to have been banished (Bell); (Robinson, (1979, pp. 3-11).

assumptions have been rigorously pursued and developed into their own consistent mathematical structures.[8]

2.3 The Perspective of Physics

Physics demands a different approach to the infinitely small than mathematics. In the abstract field of mathematics, one can develop a mathematical system based on consistent axioms, whether or not these axioms are in line with the true nature of the universe. Physics, however, is concerned with the true nature of matter. Is it infinitely divisible, as Aristotle maintained, or were the atomists correct in positing indivisible atoms from which all matter is composed? In the glossary of his book, “A Brief History of Time,” Stephen Hawking formulates the following definition for an *elementary particle*: “A particle that, it is believed, cannot be subdivided” (Hawking, 1988, p. 184). The puzzling word “believed” in this definition reveals an interesting reality. Even the greatest scientists cannot truly determine if a particle is indivisible or not. Even if a particle appears to us to be indivisible, perhaps this is only an illusion due to the inability of technology to detect its components. This obstacle prevents man from fully resolving the dilemma of the infinitely small in the area of physics.

Despite the inability to ultimately prove the existence of elementary particles, there is a rich history of scientific research in seeking such particles and studying their properties. The argument between the two schools of thought was not finally settled[9] in favor of the atomists until 1905, when Albert Einstein provided evidence for the existence of atoms (Hawking, 1988, pp. 63-66). However, scientists suspected that these atoms, too, were divisible. From Rutherford’s discovery of protons and electrons in 1910, to the discovery of the neutron in 1932, to the Stanford Linear Accelerator Center’s uncovering of quarks, the search for smaller and smaller elementary particles has come a long way. By the 1960s, particle accelerators were finding large numbers of new particles every year (Wolfram, 2002, p. 1043). The current assumption is that there are 24 elementary particles (Greene, 1999, p. 9).[10]

The question that presents itself is: “How big are these elementary particles?” The strange, but widely accepted answer is that all of the basic particles in the standard model are assumed to be *point particles* (Wolfram, 2002, p. 1043). That is, they are assumed to have zero intrinsic spatial size. This is a notion akin to the

8 A thorough reference for the historical development of the calculus is: Bos et al. (1980).
9 Or, so it seems.
10 Twelve known particles, plus an anti-particle corresponding to each one, yield a total of twenty-four.

mathematical infinitesimal, an existence to which no quantity can be assigned. Although this is the general assumption, in the 1980s, superstring theory suggested that if the presumed point-particles of the standard model could be examined with a precision significantly beyond our present capacity, each would be seen to be made up of a single, tiny, oscillating, one-dimensional loop of string (Greene, 1999, pp. 141-42). Even if this is correct, there is no way to say if these strings are the end of the line. The ultimate pursuit of the infinitely small seems to be a problem that will always elude man's grasp, yet continually engage his scientific pursuits.[11]

3. THE INFINITELY SMALL IN JEWISH LITERATURE

In order to prepare the way for a new interpretation of a halakhic debate based on an understanding of the infinitely small, this section illustrates that Jewish scholars were quite familiar with the quandary regarding the infinitely small, and discussed it both in philosophical and halakhic writings. One of the earliest explicit discussions of this issue in Jewish literature can be found in Sa'adiah Gaon's[12] *Beliefs and Opinions*. In discussing theories and proofs for creation, he references other philosophers who have put forth the theory of atoms, and the doctrine of the infinite divisibility of matter. He rejects the concept of an indivisible atom as being impossible to conceive (Sa'adiah Gaon, 1976, pp. 51-52), and the infinite divisibility of matter as being a theoretical construct, with no grounding in concrete reality (Sa'adiah Gaon, 1976, p. 45). In Book 1, Chapter 73 of his *Guide to the Perplexed*, Maimonides[13] discusses the twelve propositions common to all *mutakallemim*, a group of Arabic philosophers. The first such proposition is "The Universe, that is, everything contained in it, is composed of very small parts [atoms] that are indivisible on account of their smallness; such an atom has no magnitude...." Maimonides vehemently rejects this proposition, and says that adopting it would be tantamount to a rejection of all that has been proved in geometry. He seems to align himself with Aristotle's proposition of the infinite divisibility of matter. Thus, the inquiry into the infinitely small was certainly a topic studied by early Jewish philosophers.

11 For further study into the applicability of infinitesimals to the study of physics, see Laugwitz (1984).

12 Sa'adiah Gaon (c. 1000) was one of the last of the *geonim* in Babylonia, and was famous for his philosophical work, "Emunot V'Deot" ("Beliefs and Opinions").

13 Rabbi Moshe ben Maimon (the Rambam) (1135-1204) is known as the greatest Jewish philosopher and codifier of Jewish law in history.

Apart from the appearance of the infinitely small in Jewish philosophy, this notion is also found in the Jewish legal system. The first halakhic source to explicitly mention ideas regarding the infinitely small in the context of the Jewish legal system is the Maggid Mishneh,[14] in his answer to the *Ravad*'s question on the Rambam in *Hilkhot Shabbat* 17:12 (Kasher, 1959, p. 185).[15] Another Jewish scholar who invoked ideas involving the infinitely small to explain Jewish philosophy and law is Rabbi Joseph Rosen, the Rogatchover Gaon.[16] *Mefaneah Zephunoth* (Kasher, 1959, pp. 81-86 and 185-89) cites many instances where the Rogatchover Gaon uncovered *Hazal*'s analysis of the dilemma regarding the indivisibility versus infinite divisibility of matter. Rabbi Kasher explains that the Rogatchover's position was that *Hazal* sided with the theory of the atomists as opposed to Aristotle's doctrine of the infinite divisibility of matter.

The notion of the calculus and the infinitely small can also be found in Rabbi Yosef Dov Solevetchik's (the Rav[17]) insightful lecture on "Day and Night" (Solevetchik, 2002, p. 127). The Rav explains that Jewish law utilizes two different notions of day. One notion is day defined by light, and the other is day defined by the sun. He explains that the day defined by the sun is easy to quantify – it begins at sunrise and ends at sunset. Whenever Halakhah needs a formal quantitative definition (a *shiur*) for "day," it uses this definition.[18] On the other hand, the day defined by light presents difficulties in quantifying its starting point. The emergence of light is a continuous process that begins at dawn and can only be thoroughly

14 The Maggid Mishneh (Rabbi Vidal di Tolosa, 14th century) provides a thorough explanation of Maimonides' rulings, tracing them back to their sources, explaining the reasons for his decisions, and defending them against critics, especially the *Ravad*. Occasionally, Rabbi Vidal expresses disagreement with Maimonides.

15 The issue at hand regards the fact that one can use a *lechi* made of *avoda zara*, even though the obligation to burn *avoda zara* usually renders an object lacking in quantity (*shiur*). A *lechi*, however, can be infinitely small, a *kol shehu*, and is therefore not disqualified by its lack of quantity.

16 The Rogatchover Gaon, Yosef Rosen (1858-1936), also known by the name of his main work, *Tzafnath Paneach*, was one of the prominent talmudic scholars of the early 20th century. He was known as a "Gaon" (great sage) because of his photographic memory and sharp mind.

17 Joseph Ber (Yosef Dov) Soloveitchik(1903-93) was an Orthodox rabbi, talmudist and modern Jewish philosopher. Over the course of almost half a century, he ordained close to 2,000 rabbis who took positions in Orthodox synagogues across America. He served as an advisor, guide, mentor, and role model for tens of thousands of Modern Orthodox Jews as their favorite talmudical scholar and religious leader.

18 For instance, in calculating *zmanim* for *tefilla*, the opinion of the Vilna Gaon is that we calculate the hours of the day from sunrise to sunset.

examined through the study of calculus. This type of process is beyond the scope of a *shiur*. However, the Rav explains, when it comes to realms that do not involve a *shiur*, it is this definition of day that is selected, and dawn represents the beginning of day.[19] Although the formal mathematical system of the calculus was not invented until the 17th century, the Rav felt comfortable answering many difficulties in the Talmud and its commentaries with an idea involving the calculus. This is because the conceptual underpinnings of the calculus and the study of the infinitely small were investigated by Jewish scholars far before the formalization of the rigorous discipline known as the calculus.

4. THE APPLICABILITY OF THE INFINITELY SMALL TO THE LAW OF NULLIFICATION

As has been demonstrated, Jewish scholars were well aware of the debate regarding the infinitely small, and took positions on the matter in both philosophical and halakhic discussions. This section presents another topic that can be explained by a thorough analysis of how Halakhah deals with the infinitely small.

A major topic in Jewish law is that of dietary laws.[20] These laws can be classified into two distinct categories. The first category is those laws that prohibit certain foods from being eaten under any circumstances. The most well known among these is the restriction against eating pig. The second category is a restriction on the consumption of foods that are ordinarily permitted, but which are restricted under certain conditions. The prime example of this is the prohibition of eating meat and milk together. Meat is an entirely permitted food, as is milk. Their combination, however, creates a restriction.

19 For instance, the earliest time at which *mitzvot* of the day can be performed is at dawn, in line with the definition of day determined by light.

20 At the outset, one may object to the consideration of the Jewish dietary laws as a part of the Jewish *legal system*. However, some insight into Jewish law validates this supposition. Firstly, the *Beit Din*, the same Jewish court that judges monetary and capital matters, is also given jurisdiction in the sphere of maintaining the dietary laws. They invoke measures to prevent the violation of these laws and execute punishments on their violation. Additionally, certain prohibited foods are also prohibited from attaining any benefit. Mixtures of meat and milk are an example of such a food. Thus, such foods have absolutely no monetary value to a Jew; he is even forbidden to sell them. As such, if they are used in monetary dealings, those dealings are rendered invalid, as the Jewish legal system considers these foods worthless. Thus, laws that gain their existence from the sphere of dietary laws actually make their appearance in the monetary sphere as well. Therefore, these laws can fairly be subsumed under the context of the Jewish legal system.

The Jewish dietary laws extend beyond the realm of eating prohibited foods by themselves, and into the realm of dealing with mixtures of permitted and prohibited foods. For example, what is the law if somebody is cooking a pot of meat, and accidentally drops in a piece of pork? Does the entire pot of meat become forbidden? The answer is twofold. Firstly, if the piece of pork is still identifiable, it must be removed before the stew is consumed. In the event that the pork is dissolved and unidentifiable, then the ruling would depend on the quantity of the pork, i.e. whether the pork imparts its flavor into the meat. If the taste of pork is manifest in the stew, then the entire stew is prohibited. If, however, the amount of pork is too minimal to make any noticeable impact on the stew, then the *Law of Nullification* treats the pork as nonexistent and the stew can be eaten.

The keen observer may notice an intrinsic problem with the above system. How is one to know if in fact the pork imparts taste or not? If a person tastes it and finds that it does have the taste of pork, then isn't he already in violation? The simplest solution to this problem is to have a gentile, a person who is not prohibited from eating pork, taste it and report if the taste of pork is present. If this option is not available, then the law is as follows: If the pork is less than 1/60th of the mixture, then one has the right to assume that it does not impart taste and its presence is therefore nullified. But, if the pork is a more significant portion of the mixture than 1/60th, then the assumption is that the taste of pork is manifest and the entire mixture becomes prohibited. The same Law of Nullification of small quantities applies regarding the combination of milk and meat. If a drop of milk falls into a pot of meat, then it depends again on the quantity of the milk. If the milk is less than 1/60th of the stew, then the milk is nullified and the stew is permitted. However, if the quantity of milk is more than 1/60th, then the stew is prohibited.[21]

The Talmud[22] in Tractate Pesahim on folio 76a presents an interesting case. If a person is cooking a pot of milk and a large[23] piece of meat drops in, can he simply remove the piece of meat, and eat the meat and milk separately? Or, is there a

21 Although the presence of taste may vary from food to food, in the interest of practicality the Jewish legal system fixes a quantity at which we could safely assume that there is no taste. For further discussion of this principle, see *Shulhan Arukh*, Section Yorah Deah, Chapter 98, Paragraph 1.

22 The Babylonian Talmud, referred to as the Talmud, is a major Hebrew work that represents the culmination of centuries of knowledge, which was formerly transmitted orally. The Talmud is broken up into a number of tractates (or treatises). Each tractate is divided into folios, each of which consists of two pages. A good reference to the Talmud is *Encyclopaedia Judaica*, 1972, 15: 750-79.

23 Large enough that the meat is more than 1/60th of the milk, and its flavor will not be nullified.

concern that the meat imparted flavor to the milk and vice versa? The Talmud concludes that it depends on the temperatures of the meat and the milk. If they are both *hot*,[24] then the heat will be a vehicle of mutual transfer of flavor and they are both prohibited. At the other extreme, if they are both *cold*,[25] one could simply remove the meat, wash it off, and then eat them both. There is no vehicle of transference of flavor. The more interesting cases are when there is conflict, when one is *hot* and the other is *cold*. Which one dominates? This is disputed, and the Talmud concludes like the opinion of Shmuel,[26] who maintains that the pot of milk that is on the bottom dominates. Namely, if the milk is *hot* and the meat is *cold*, then one treats it as if the milk heats up the meat and the taste thereby transfers. Consequently, both are prohibited. If, however, the milk is *cold* and the meat is *hot*, then it can be treated as if the milk cools down the meat, in which case there is no transference. Both foods are then permitted.

After Shmuel's ruling is presented, the Talmud proceeds to question his last statement. Even if the *cold* milk will cool down the *hot* meat, this cooling process certainly cannot be instantaneous. Thus, during the cooling process, it is virtually impossible that there will not be a transfer of some milk, albeit a very small quantity. How then could Shmuel permit the entire piece of meat? Won't the outer surface of the meat contain some milk absorbed in it and therefore be prohibited? Based on the force of this question, the Talmud qualifies Shmuel's ruling. Namely, even Shmuel agrees that in this case one must peel off the surface layer of the meat and discard it. This layer is essentially a mixture of meat and milk. This concludes the discussion in the Talmud.

Based on his analysis of this portion of the Talmud, the commentator *Riva*[27]

24 The legal definition of *hot* is that it is hot enough to burn the finger (approximately 50° C). Whenever the term *hot* appears in this paper, it will be in this sense.

25 *Cold* refers to any temperature below that of the legal definition of *hot*, as above. Throughout this paper, *cold* will have this connotation.

26 Shmuel was a great Babylonian Jewish scholar who lived early in the second century. He is mentioned in the Talmud hundreds of times. He had extensive knowledge of medicine and astronomy, as well as Jewish law. He was known as "Shmuel the astronomer," and declared "The paths of the heavens are as familiar to me as the streets of Nehardea (his town)" (Frieman, 1995, pp. 296-99).

27 The *Riva* (c. 1130) is an acronym in Hebrew for Rabbi Yitzchak ben Asher. He was an early commentator from a school of commentators known as *Tosafot*, literally "additions." The *Tosafot* is a running commentary that appears alongside the Talmud in most editions. It is a compilation of opinions of many students from the school of *Tosafot* and was written during the 12th and 13th centuries (*The Rishonim*, 1982, p. 129).

makes a stark point.[28] The Talmud qualifies Shmuel's ruling regarding the permission to eat the meat, but not the milk. Why should there be a difference? During the cooling process, isn't it true that the surface of the milk will absorb a minimal amount of meat, just as the surface of the meat absorbed a minimal amount of milk? What can be done about this? One cannot peel off the surface of the milk as was done to the meat, because the surface is not fixed. It became mixed up with the rest of the milk. Thus, a small amount of prohibited milk is mixed in with the regular milk. The *Riva*, therefore, follows Shmuel's line of reasoning and concludes that one must treat such a case as any mixture is treated. If the amount of milk from the surface is greater than 1/60th of the rest of the milk, then the entire quantity of the milk is prohibited. If, however, it is less than 1/60th, then it is nullified. The *Riva*'s qualification seems unassailable.

Rabbeinu Tam,[29] another talmudic commentator, surprisingly disagrees.[30] He maintains that since it is not possible to peel off the surface of the milk, as in the case of the meat, the milk is simply permitted. For some reason, he does not demand the nullification of the surface quantity of the milk. This position presents a major difficulty. It seems clear that the surface of the milk does in fact absorb the flavor of the meat and becomes prohibited. Why, then, is one not required to nullify it? If, for instance, a large piece of meat falls onto a spoonful of milk, the surface of the milk will certainly be more than 1/60th of the spoonful. The milk should be prohibited as in any case of a mixture of prohibited and permitted items. The position of Rabbeinu Tam seems untenable and inconsistent with the ordinary dietary laws.

Despite its apparent difficulty, the position of Rabbeinu Tam could be explained based on a deep understanding of how the Jewish legal system treats the infinitely small. In order to appreciate this concept, the following analysis could be suggested. In the abovementioned case, one is required to peel off the surface of the meat. If one were to ask, "How deep is the surface?" what would the answer be? How much must be peeled away? Could a knife that is 1 mm. thick be used to peel away the surface? Will it get the entire surface? The answer is decidedly, "yes." Let's say

28 The comments of *Riva* and Rabbeinu Tam appear in *Tosafot*, on the side of the Talmud in the paragraph beginning with "Tanya Eidach."

29 Rabbeinu Tam's (1100-71) real name was Rabbi Yaakov ben Meir. The word "Tam" in Hebrew means wholesome. He was called Tam after his namesake, Jacob. The Bible (Genesis 25:27) calls Jacob an "Ish Tam," a wholesome man, for his completeness of character. Rabbeinu Tam was the head of the French school of Tosafists and is considered by many as the greatest talmudist of his generation (*The Rishonim*, 1982, p. 127).

30 In some places this opinion is quoted in the name of the Tosafist Rabbi Yitzhak of Dampierre (c. 1120-c. 1200), known as "the Ri," instead of in the name of Rabbeinu Tam.

the knife is 0.1 mm,? 0.01mm? 0.001 mm.? "Yes. Yes. And, Yes." The truth is that the surface is an unquantifiable entity. Any degree of penetration into the meat brings one beyond the surface and into the inside. The surface is a qualitative entity that is infinitely small. All one could posit about the thickness of the surface is that its limit is 0. Despite the fact that this surface is infinitely small, it does exist (just as an infinitesimal exists in mathematics, and a point particle exists in physics, though they are smaller than any fixed quantity).[31] Thus, one cannot simply bite into the piece of meat, since its surface is prohibited. No amount of prohibited food should be eaten. The situation is therefore analogous to the case of a piece of pork in a pot of meat. Despite the fact that the pork is less than 1/60th of the stew, one must remove it from the stew if it is identifiable. The same is true with the surface. Since it is identifiable, one must peel it off. Whatever quantity one peels off will be enough, and will certainly contain the infinitely small surface. This explains the logic of the ruling regarding the meat.

With this insight into the law of removing the surface of the meat, one is in a position to understand the position of Rabbeinu Tam regarding the milk. As the meat falls into the milk, the infinitely small surface of the milk absorbs the meat. However, this surface becomes unidentifiable from the rest of the milk. Thus, the case is analogous to the case of an unidentifiable piece of pork in a pot of meat. How does one proceed? As above, one must compare the quantities of the pork and the meat. If the pork is more than 1/60th of the stew, the stew is prohibited. If it is less, it is permitted. One must follow the exact same reasoning in our case and see if the surface is more than 1/60th of the milk or less. The size of the surface is infinitely small. Thus, 60 times the surface is also infinitely small. It follows that the surface of the milk is certainly less than 1/60th of the milk itself, no matter how small a quantity of milk one has. The surface will *always* be nullified by the milk

31 One may be perplexed how it is that only an infinitely small amount of milk is absorbed in the meat. After all, practically, some quantifiable amount must have been absorbed. The answer to this question is as follows: Since we are dealing with a case of *hot* meat into *cold* milk, we follow Shmuel's position that we have the full right to assume that the milk will cool down the meat and prevent transference. However, since it is physically impossible that during the cooling process itself there is *no* transfer, the ruling cannot deny a practical fact. The entire piece of meat cannot be permitted. However, all we know is that *some* transference must have taken place. There is no known quantity that transferred. The result is that as long as we peel off *any* amount off of the surface of the meat and discard it, we are no longer denying the fact of transference. It is factually possible that the absorption was less than the peel that we removed. We can assume that any fixed quantity is more than the quantity that was absorbed. It is in this sense that we postulate that the prohibited amount is infinitely small.

in which it is contained. Even 0.0000001 is more than sixty times as large as an infinitely small entity. Thus, Rabbeinu Tam's proposition that the milk will be permitted under any circumstances is understandable. Once one gains insight into the nature of the infinitely small, it becomes clear that the prohibited milk will always be nullified.[32]

Although it seems that the problem has been answered, a tremendous difficulty once again arises. The answer seems too cogent. How could the *Riva*, an equally great expert in Jewish law and abstract analysis, contest this point? How could he demand that one has 60 times the surface, if the surface is not truly quantifiable? The explanation could be as follows. The *Riva* understood the abstract point of Rabbeinu Tam, but had a different position regarding the incorporation of the infinitely small into a practical legal system. He recognizes that, in theory, the surface is infinitely small. However, in practice, there is no such thing as an infinitesimal. There is no knife whose thickness is infinitely small. Practically, there is a size to the thickness of the surface that man can peel off of the meat. Whatever quantity one would get by man's smallest peel, must be measured, and must be less than 1/60th of the mixture.[33] Thus, it is possible that the surface quantity of the milk will not be nullified. The *Riva* realizes that in a theoretical world one can continuously divide a substance. Yet, at the same time, he recognizes that man's world is a world of discrete quantities. It is therefore a world in which successive division results in "indivisibles." It is these "indivisibles" (which have quantity, albeit small) which are in the spoon of milk and resist nullification.

To summarize, the point of contention between Rabbeinu Tam and the *Riva* revolves around the same point that was so elusive to the world's mathematicians,

32 For a similar understanding of Rabbeinu Tam's position, see the "Chazon Ish" (Karelitz, 1991, 9: 6). Rabbi Avrohom Yeshaya Karelitz (1878-1953) devoted his life to the study of Torah, although he also learned sciences such as astronomy, anatomy, and mathematics, since he felt that a knowledge of these subjects was necessary for a full understanding of Jewish law and practice (*Judaism* 101). For a different approach, see *Aruch Ha'Shulchan* (Epstein, 1987, 91:16-17). Rav Yechiel Michel Epstein's (1829-88) major work, *Aruch Ha'Shulchan*, provides an in-depth analysis of many facets of Jewish law.

33 One may ask how we evaluate the surface quantity of the surface of the milk. The Yad Avraham, in his commentary on the *Shulhan Arukh*, Section Yorah Deah, Chapter 91, paragraph "Vi'Hachalav..." (*Shulhan Arukh*, 1977) gives an insightful answer. He notes that we are not trying to evaluate the entire surface of the milk, but only that portion that came into contact with the meat. This portion, however, will be equivalent to the surface of the meat itself. Therefore, in order to evaluate the surface of the milk, we could simply peel off our thinnest layer from the surface of the meat and measure it, and that quantity will correspond to the quantity of milk we are looking for.

philosophers and physicists. Namely, what is the proper way to understand the infinitely small? More specifically, how will the Jewish legal system codify a method for dealing with the infinitely small? Rabbeinu Tam maintains that it will utilize the doctrine of unlimited indivisibility that recognizes the existence of a substance whose limit in size is zero. His approach is in consonance with the modern approach of Cauchy, which recognizes an ever-shrinking quantity, and deals with it through the concept of a limit.[34] *Riva*, on the other hand, maintains that the practical legal system must build on the supposition of the existence of "indivisibles." In this regard, his position is akin to Newton, Leibnitz, and the other mathematicians who preceded Cauchy. They all treated the infinitely small as a quantity, in order to deal with it in a concrete manner. It is interesting to note that Rabbeinu Tam and the *Riva* preceded Newton, Leibnitz and Cauchy by hundreds of years.[35] Although their analysis was not in the realm of mathematics itself, it shares with the calculus this major feat – they both create a concrete system that readily handles the infinitely small.[36] [37]

34 Another instance of Rabbeinu Tam's recognition of the idea of an infinitely small substance whose limit is zero can be found in Tractate Pesahim on Folio 26b in *Tosafot*, paragraph beginning "Chadash Yutzan." The Gemara discusses a case where bread is baked using wood from which it is forbidden to derive benefit. If the bread was baked with coals derived from this wood, the bread would be permitted. This is because the coals are considered a new substance, and therefore, at the time the bread is baked, the wood is no longer extant. However, if the wood itself bakes the bread, before it becomes coals, then there is a dispute between Rebbi and the Rabanan if the prohibition will transfer to the bread or not. *Tosafot* explains the dispute as follows: Rebbi's position is straightforward – since the wood directly bakes the bread, the prohibition will transfer from the wood to the bread. The Rabanan, however, argue that it is not the extant wood that bakes the bread, but the *Ma'Shehu Ha'Nisraph*, the infinitely small surface of the wood that was burnt. This surface is not defined as an extant substance as it lacks quantity, and is therefore akin to the coals that do not prohibit the bread. This explanation of *Tosafot* was presented by Rabbi Yisrael Chait (oral communication, 2005).

35 Furthermore, Rabbeinu Tam and the *Riva* were explaining the view of Shmuel, who lived in the second century.

36 For other instances of rabbinical scholars keeping pace with the mathematical community of their time, see Garber and Tsaban (1998), pp. 75-84; (2001), pp. 10-15.

37 The difficulty that a legal system faces in concretizing a method of dealing with the infinitely small can be found in the American legal system as well. One instance of this is "the zero tolerance policy" adopted by many states in America. This policy involves the amount of alcohol a minor is allowed to have in his blood while driving. For adults, there is a legal amount allowed before one is in violation of driving under the influence. However, for minors, there is zero tolerance – driving with any amount of alcohol in one's blood is an offense. Although this is the spirit of the law, the precise formulation of the law in different states is perplexing.

5. CONCLUSION

Thus far in the history of man's intellectual and scientific pursuits, the quest to penetrate the depths of the infinitely small has yielded great dividends. The grandest of the contributions of this study to the secular world is the foundation of the

The relevant text in Texas State Senate Bill 35, Sec. 106.041 states that a violation is committed when driving with "any *detectable* amount of alcohol in the minor's system." It is interesting to note that the text of the bill does not truly point to *zero* tolerance. It actually tolerates an undetectable amount. Even if a minor drinks an amount that is not detectable, he has still not drunken zero. He is still drinking and driving. Why, then, does the text of a zero tolerance law actually tolerate an undetectable amount of alcohol? Why did the authors of the law include the word "detectable"? The answer is clear. Although the theoretical idea behind the law is to create zero tolerance, such a law is completely impractical. An infinitesimal amount of alcohol is greater than zero, but is not detectable. It therefore cannot be incorporated in the corpus of the law. This is reflective of the fact that a legal system has its own practical concerns in dealing with the infinitely small and zero. The legal definition of zero cannot be equivalent with the mathematician's definition of zero. This approach follows the line of thought of the *Riva* in formulating Jewish law.

Despite the solid logic behind this explanation of the Texas state law, one is confounded on investigating the corresponding law in the state of Pennsylvania. Section 3718 of the Pennsylvania Vehicle Code states that a violation is committed when a minor drives "while having *any* alcohol in his system." It is clear that the writers of the Pennsylvania State Law had a different point of view than those of Texas. Despite the fact that the practicalities of a legal system may limit enforcement to detectable amounts of alcohol, the formulation of the zero tolerance law itself must allow zero tolerance. *Any* amount of alcohol is a violation, however small this amount may be. The state law of Pennsylvania is more in consonance with Rabbeinu Tam's approach to a legal system. The law can and must recognize the theoretical, mathematical concept of zero in its formulation of laws. The concept of the law must not be corrupted because of the practical difficulties that it may bring forth.

A practical difference between the two state laws would arise in the following scenario: Let's assume that a police officer witnesses a minor drinking a teaspoon of alcohol, and driving. Would the law be able to prosecute this minor? In the state of Texas, it would seem not. Since a teaspoon is undetectable, no violation has occurred. The state of Pennsylvania, however, could press charges. A teaspoon is not zero and is, therefore, in violation of their state law. Both points of view are valid approaches to the formalization of a legal system involving quantities immeasurably close to zero. The dilemma of dealing with the infinitely small in the context of law provides another facet of this ubiquitous quandary, which continually confronts man in every sphere of study.

For another application of the dilemma of including infinitesimals in American law, see Veilleux (1992). This paper discusses at length the minimum quantity necessary for prosecuting a violation of criminal "possession" of drugs. The term "possession" conceptually has no minimum, but one finds difficulty creating a practical law involving immeasurable substances. This paper discusses the different state laws and all the cases and legal literature relating to this topic. It is therefore a good reference to study the scope of the applicability of the infinitesimal to the American legal system.

calculus, the most powerful modern tool in probing the mysteries of the universe. This endeavor has also been a fundamental arena for man's philosophical inquiries and an unrelenting source of investigation in the realm of particle physics. This area of analysis is of fundamental significance in Jewish literature as well. Jewish philosophers have long pondered the solution to the dilemma regarding the infinitely small in the universe. The halakhic system, too, contains many topics that can only be fully appreciated by an understanding of the Jewish scholars' opinion on the proper application of the infinitely small to a legal system such as Halakhah. In their abstract analysis of the infinitely small, one should not be surprised to find that Jewish scholars are at least on a par with the great thinkers of the secular world.

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